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DESIGN OF FULL-ORDER OBSERVERS FOR SYSTEMS WITH UNKNOWN INPUTS BY USING THE EIGENSTRUCTURE ASSIGNMENT

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ABSTRACT

In this paper a full-order observer is suggested in order to achieve finite-time reconstruction of the state vector for a class of linear systems with unknown inputs. The proposed design procedure is a combination of the approaches proposed by Lin & Wang [1] and Trinh & Ha [2]. The resulted observer has been improved, from the robustness point of view, by this paper's authors by using a novel and efficient method; it consists of adding three robustness terms which cancel the negative effect of the uncertainties which can appear in the system. The effectiveness of the suggested design algorithm is illustrated by a numerical example (aircraft lateral motion).

Key Words: Observer, unknown input, algorithm, aircraft motion.

I. INTRODUCTION

1.1 Antecedents and motivations

The plant input and output signals are used to estimate the plant state, which is then employed to close the control loop [3,4]. The aim of observers is to augment or replace sensors in a control system. Starting from the first observers, introduced by Luenberger [5,6], the observers for plants with both known and unknown inputs have been developed resulting in the so-called unknown input observer architectures, such as, for example, those in [7–12]. A physical process is often subjected to disturbances which have as origin the noises due to its environment, uncertainty of measurements, fault of sensors or actuators. These disturbances affect the normal behavior of the process and the estimation of these disturbances is needed in order to conceive a control strategy able to minimize their effects. Disturbances are called unknown inputs if they affect the process input, their presence making difficult the state estimation [13]. The state estimation problem for linear multivariable system, subjected to unknown inputs, has received considerable attention in recent decades [10,14–18]. The dimension of the observer is considerably increased in [13] and that is why the approach of Wang *et al*. [14] is more interesting; they proposed a method to design reduced-order observers without any knowledge of these inputs; existence conditions for this observer have been provided by Kudva *et al*. [15]. Silverman's inverse method [17], the generalized inverse matrix, and the singular value decomposition are useful in linear observers' design process [10]. There are two categories of papers describing observer design methods: the first one supposes *a priori* knowledge of information on these non-measurable inputs, while the second category proceeds either by estimation of the unknown inputs, or by their complete elimination from the equations of the system [13].

If a system is affected by unknown inputs, in order to reconstruct the entire state vector, strict constraints must be fulfilled [4]. Full-order observers for linear systems with unknown inputs were designed by Valcher in [19]; here, the authors presented the necessary and sufficient conditions for the existence of such observers. These conditions are sometimes difficult to meet and, that is why, Valcher's work has been continued and improved in [20], where the constraints of the observers have been relaxed allowing delays in the observer, but no design algorithm was provided. By using a higher dimensional system which incorporated the delayed states of the system into a new state vector, in [21] a new delayed observer has been designed. Although this paper provides the observer geometric existence conditions, the drawback of the approach presented in [21] is related to the dimension of the observer which is higher than the system dimension [21–24].

We may conclude that there are a lot of observers for linear systems with unknown inputs [4]. There are three important design methods: geometrical methods (introduced first time by Bhattacharyya [16]), algebraic methods (used in observer design by Kudva *et al*. [15], Chadli *et al*. [25], Hou & Muller [26], Darouach *et al*. [10], Koenig *et al*. [27], and Trinh *et al*. [28]), and methods that use the generalized inverse [29]. Each of them has advantages and disadvantages.

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1.2 Main contribution

The classical full-order observers are easier to implement from the software point of view, their disadvantages being related to the important number of constraints [4]. This paper presents a new full-order observer design for the state estimation of the systems with unknown inputs. It will be shown that the problem of full-order observers for linear systems with unknown inputs can be reduced to a standard one (unknown inputs do not interfere in the observer equations). The observer existence conditions are also given.

The new observer is based on two interesting observer design approaches suggested by Lin & Wang [1] and Trinh & Ha [2]. In the first one, a simple and direct method for full-order unknown input observer design is proposed; the procedure is straightforward and no further complex conditions are needed. The eigenstructure assignment is used for the determination of the observer matrices [1]. The main disadvantage of the paper is that some of the matrices are chosen randomly and there is no guarantee and proof that these matrices lead to a convergent observer with good convergent speed. In order to overcome this drawback, the authors of the present paper used the method presented by Trinh & Ha in [2] for the design of a reduced-order linear functional state observer in the case of linear systems with unknown inputs; thus, we obtained a full-order observer for the state estimation in the case of systems with unknown inputs. In order to extend the method presented in [2] to the general class of unknown input observers, we modified this method and we adapted it to our case; in our case, this method represents now a subroutine in our new observer design procedure. On the other hand, the method presented in [1] to improve the robustness of the observer with respect to the eigenvalues' sensitivity and uncertainties which can appear in the system's dynamics is difficult and it is replaced by a novel method which uses the Lyapunov theory to calculate three robustness terms whose aim is to cancel the negative effect of the system's uncertain matrices (if these exist). Using these robustness terms, the choice of the observer eigenvalues, and hence the determination of the system's eigenvectors (occurring in the observer matrices' determination procedure), is randomly achieved without affecting the observer robustness. In [2], the possible system's uncertainties are not taken into account, this being one of the paper disadvantages; this drawback is overcome here.

II. OBSERVER DESIGN PROCEDURE

2.1 Problem statement

In this section of the paper we design a new observer for the state estimation problem in the case of linear time-variant systems with unknown inputs. The new design procedure represents a mixture of the design approaches presented in [1] and [2]; moreover, to cancel the possible uncertainties (small variances of the system's matrices) and to improve the design approach of the observer, some robustness terms and new calculation techniques will be added in the observer design. In [1] and [2] simple and direct observer design methodologies are presented for the systems having unknown inputs; the algorithms have some disadvantages and the paper's authors will cancel them in order to obtain a more robust observer.

Consider the following linear dynamic system [2]:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dv(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

where $x \in \mathcal{R}^n$ is the system state vector, $u \in \mathcal{R}^m$ – the system known input vector, $v \in \mathcal{R}^q$ – the system unknown input vector, and $y \in \mathcal{R}^r$ – the output vector; the known matrices A, B, C, D have appropriate dimensions: $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{r\times n}$, $D \in \mathcal{R}^{n\times q}$. We assume that $\text{rank}(D) = q$, $\text{rank}(C) = r$, and matrix C may be brought to the canonical form $C = [C_1 \ 0_{r\times(n-r)}]$, where $C_1 \in \mathcal{R}^{r\times r}$ is a full rank matrix; any matrix C may have the previous canonical form by using an orthogonal transformation [2].

The observer for estimating the state vector is chosen as follows [1]:

$$\begin{cases} \dot{z}(t) = Qz(t) + Ly(t) + Gu(t), \\ x(t) = z(t) + Sy(t), \end{cases} \tag{2}$$

where $z \in \mathcal{R}^n$, $\hat{x} \in \mathcal{R}^n$, $Q \in \mathcal{R}^{n\times n}$, $L \in \mathcal{R}^{n\times r}$, $G \in \mathcal{R}^{n\times m}$, $S \in \mathcal{R}^{n\times r}$. Let $e(t) = x(t) - \hat{x}(t)$ be the error vector of the observer. In [1] it is proved that the observer dynamics is:

$$\dot{e}(t) = Qe(t) \tag{3}$$

if and only if the following conditions are fulfilled [1]:

$$\begin{aligned} &LC + SCA - QSC - A + Q = 0, G + SCB - B = 0, \\ &SCD - D = 0. \end{aligned} \tag{4}$$

The dynamics (3) corresponds to a system with asymptotically convergent error if Q is Hurwitz; this is because the solution of (3) is $e(t) = e_0 \ \exp(Qt) = e(0) \exp(Qt)$ and, regardless of the e_0 value, if Q is Hurwitz ($Q < 0$), it results $\lim_{t\to\infty} e(t) = 0$ [7, 23]. Later, it will be shown that Q is Hurwitz and $\lim_{t\to\infty} e(t) = 0$ if conditions (4) are fullfilled. Thus, the problem to solve now is to determine the S, L, G, and the stable matrix Q satisfying equations (4).

2.2 Design of the full-order observer

For the determination of the observer unknown matrices, in [1], the technique of pole placement (eigenstructure assignment) is used; thus, we denote with $\lambda_i, i=\overline{1,n}$ – the eigenvalues of the matrix Q and with $h_i, i=\overline{1,n}$ – the left eigenvectors which correspond to λ_i.

The right eigenvectors for the matrix Q are column vectors satisfying the equation $Qv_i = \lambda_i v_i,\ i=\overline{1,n}$. This equation can also be written under the matriceal form $Q\tilde{R} = \tilde{R}\Lambda$, where $\tilde{R}$ is a matrix whose columns are the right eigenvectors and $\Lambda = \mathrm{diag}(\lambda_i)$. The left eigenvectors (denoted above with h_i) are row vectors satisfying equation $h_iQ = \lambda_i h_i,\ i=\overline{1,n}$ or the matriceal equation $HQ = \Lambda H$. The left eigenvalues are the same with the right eigenvalues, while the right eigenvectors (v_i) are different from the left eigenvalues h_i. The relation between the matrix whose columns are the right eigenvectors (matrix $\tilde{R}$) and the matrix having as rows the left eigenvectors (matrix H) can be easily deduced starting from the equation $Q\tilde{R} = \tilde{R}\Lambda$, which left multiplied by $\tilde{R}^{-1}$ leads to $\tilde{R}^{-1}Q\tilde{R} = \Lambda$. By right multiplication of the previous equation by $\tilde{R}^{-1}$, we get $\tilde{R}^{-1}Q = \Lambda\tilde{R}^{-1}$. Taking into account the equation $HQ = \Lambda H$, it yields $H\tilde{R}^{-1} = I$. From this, we obtain:

$$h_i v_j = \begin{cases} 1, i = j, \\ 0, i \neq j, \end{cases} \tag{5}$$

and the conclusion $h_i \perp v_j,\ i \neq j$ (h_i and v_j are orthogonal).

The choice of the eigenvalues for the matrix Q is closely related to the elements of the matrix Q; thus, if Q is square with real elements, according to Perron-Frobenius theorem, there exists a positive number $\tilde{r}$ – Perron-Frobenius eigenvalue such that $\tilde{r}$ is an eigenvalue of the matrix Q and the module of any other eigenvalue of matrix Q is strictly smaller than $\tilde{r}$. This eigenvalue satisfies the inequality [30]:

$$\min_i \sum_j q_{ij} \le \tilde{r} \le \max_i \sum_j q_{ij}, \tag{6}$$

where $q_{ij}, i, j = \overline{1,n}$ are the elements of matrix Q. Because the eigenvectors h_i are linearly independent, the first and the third condition of (4) lead to the equations:

$$h_i(LC + SCA - QSC - A + Q) = 0, h_i(SCD - D) = 0. \tag{7}$$

By using the notations [1]:

$$\overline{s}_i = h_i S, \overline{l}_i = h_i L, \tag{8}$$

the equations (7) become:

$$\begin{bmatrix} h_i & \overline{s}_i & \overline{l}_i \end{bmatrix} \begin{bmatrix} -(A-\lambda_i I) & -D \\ C(A-\lambda_i I) & CD \\ C & 0 \end{bmatrix} = 0, i = \overline{1,n}. \tag{9}$$

Moreover, by using the notations [1]:

$$H = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}_{n\times n}, \overline{S} = \begin{bmatrix} \overline{s}_1 \\ \overline{s}_2 \\ \vdots \\ \overline{s}_n \end{bmatrix}_{n\times r}, \overline{L} = \begin{bmatrix} \overline{l}_1 \\ \overline{l}_2 \\ \vdots \\ \overline{l}_n \end{bmatrix}_{n\times r}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}_{n\times n}, \tag{10}$$

we obtain:

$$\overline{S} = HS, \overline{L} = HL, HQ = \Lambda H. \tag{11}$$

Theorem 1 [1]**.** If exist the full row rank matrices $M_i, N_i, P_i, i=\overline{1,n}$ with the property

$$\begin{bmatrix} M_i & N_i & P_i \end{bmatrix} \begin{bmatrix} -(A-\lambda_i I) & -D \\ C(A-\lambda_i I) & CD \\ C & 0 \end{bmatrix} = 0, \tag{12}$$

where $M_i \in \mathcal{R}^{n\times n},\ N_i, P_i \in \mathcal{R}^{n\times r}$, then

$$h_i = f_i M_i, \overline{s}_i = f_i N_i, \overline{l}_i = f_i P_i \tag{13}$$

are the solutions of (9), with $f_i \in \mathcal{R}^n$ – random vectors.

Proof [1]**.** The proof of the theorem is easily achieved by writing (12) in the form:

$$\begin{cases} -M_i(A-\lambda_i I) + N_i C(A-\lambda_i I) + P_i C = 0, \\ -M_i D + N_i CD = 0 \end{cases} \tag{14}$$

and left multiplication of the two previous equations by $f_i \in \mathcal{R}^n,\ i=\overline{1,n}$; we get [1]:

$$\begin{cases} -f_i M_i(A-\lambda_i I) + f_i N_i C(A-\lambda_i I) + f_i P_i C = 0, \\ -f_i M_i D + f_i N_i CD = 0; \end{cases} \tag{15}$$

Now using the equations (13), one gets:

$$\begin{cases} -h_i(A-\lambda_i I) + \overline{s}_i C(A-\lambda_i I) + \overline{l}_i C = 0, \\ -h_i D + \overline{s}_i CD = 0, \end{cases} \tag{16}$$

this system being equivalent to the one in (9).

2.3 Calculation of the matrices M_i, N_i, and P_i

The determination of the matrices M_i, N_i, and $P_i,\ i=\overline{1,n}$ is equivalent, after the random choice of the row vectors $f_i \in \mathcal{R}^n,\ i=\overline{1,n}$, with the determination of the row vectors h_i, $\overline{s}_i$, and $\overline{l}_i,\ i=\overline{1,n}$ and, after that, with the calculation of the matrices H, $\overline{S}$, $\overline{L}$, and Λ– equations (10). The matrices S, L, and Q can be calculated if the matrix H is non-singular (invertible) by means of equations (11) as follows [1]:

$$S = H^{-1}\bar{S}, L = H^{-1}\bar{L}, Q = H^{-1}\Lambda H. \tag{17}$$

Choosing the eigenvalues of the matrix Q situated in the left-hand side of the complex plane (Re$\{\lambda_i\} < 0$ it is evident that matrix Λ satisfies the condition: $\Lambda < 0$; on the other hand, the matrix H is calculated with respect to the left eigenvectors (h_i) which correspond to λ_i. In [1] it is shown that the formula for the calculation of matrix Q, (17), will always lead to a Hurwiz matrix and, in these conditions, it is proved that $Q < 0$ for any λ_i situated in the left-hand side of the complex plane.

The matrix G is determined with respect to matrix S by using the second equation of (4);

$$G = B - SCB = B - H^{-1}\bar{S}CB. \tag{18}$$

If rank(CD) = rank(D), the third equation of (4) has the solution [5]:

$$S = D(CD)^{+} + S_0\left\lfloor I_r - (CD)(CD)^{+}\right\rfloor, \tag{19}$$

where $S_0 \in \mathcal{R}^{n\times r}$ is a randomly chosen matrix, while $(CD)^+ = [(CD)^T CD]^{-1}(CD)^T$. If we use this method, we can easily determine the matrix S—equation (19) and then the matrix G by means of the second equation of (4), but the first equation of (4) has now two unknowns (matrices L and Q) and, therefore, it is difficult so solve; this is why, it is not judicious to use this method.

The approach presented in [1] does not include a methodology for the calculation of the matrices M_i, N_i, and P_i − solutions of the equation (12). To overcome this drawback, we introduce in our observer design approach an ingenious subroutine which is often used when designing the functional observers for the systems with unknown inputs [2]. In order to use the method from [2], first of all, we must write (12) in the form of (14), or, by means of the notations

$$\begin{aligned} &T_i = -M_i + N_i C \in \mathcal{R}^{n\times n}, E_i = A - \lambda_i I \in \mathcal{R}^{n\times n},\\ &K = -I_n \in \mathcal{R}^{n\times n}, \end{aligned} \tag{20}$$

under the equivalent form:

$$\begin{cases} T_i E_i + P_i C = 0_{n\times n},\\ T_i D = 0_{n\times q},\\ M_i = KT_i + N_i C, i = \overline{1, n}. \end{cases} \tag{21}$$

The previous system has four unknowns (M_i, N_i, P_i, T_i) and only three equations. The system's solution is made for each $i = \overline{1, n}$, because for each eigenvalue λ_i, $i = \overline{1, n}$, the matrix E_i changes. Taking this into account, for ease of notation, in the approach for solving the systems with form (21) the subscript "i" is omitted. The n systems (21) are written under the form:

$$\begin{cases} TE + PC = 0_{n\times n},\\ TD = 0_{n\times q},\\ M = KT + NC. \end{cases} \tag{22}$$

These are not the same with the ones presented in [2], the paper's authors adapting the methodology presented in [2] to the present design case. Thus, first of all, we consider:

$$\begin{aligned} M &= [m_1 \quad m_2 \quad \cdots \quad m_n] = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n}\\ m_{21} & m_{22} & \cdots & m_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}_{n\times n},\\ T &= [t_1 \quad t_2 \quad \cdots \quad t_n] = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n}\\ t_{21} & t_{22} & \cdots & t_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{bmatrix}_{n\times n}, \end{aligned} \tag{23}$$

where m_j and t_j are the columns j of the matrices M and T, respectively. The third equation (22), by using (23) and the expression of matrix C, *i.e.* $C = [C_1 \ 0_{r\times(n-r)}]$, becomes:

$$\begin{aligned} &[m_1 \quad m_2 \quad \cdots \quad m_n] - K[t_1 \quad t_2 \quad \cdots \quad t_n] - [NC_1 \quad 0_{n\times(n-r)}]\\ &\quad = 0_{n\times n}, \end{aligned} \tag{24}$$

equivalent with [2]

$$m_j = Kt_j, \ j = \overline{(r+1), n}. \tag{25}$$

Remark 1. In the design procedure, we assume that the number of system's outputs (r) is less than the number of states (n). This is not a restrictive assumption because it generally holds and because this assumption is very often valid in the design procedure of full-order observers for systems with unknown inputs.

With the notations:

$$\begin{aligned} &\tilde{t} = \left[t_{r+1}^T \quad t_{r+2}^T \quad \cdots \quad t_n^T\right]^T \in \mathcal{R}^{n(n-r)\times 1},\\ &\tilde{m} = \left[m_{r+1}^T \quad m_{r+2}^T \quad \cdots \quad m_n^T\right]^T \in \mathcal{R}^{n(n-r)\times 1},\\ &\Omega = \mathrm{diag}\{K\} = \mathrm{diag}\{-I_n\} \in \mathcal{R}^{n(n-r)\times n(n-r)}, \end{aligned} \tag{26}$$

the equations (25) lead to [2]:

$$\tilde{m} = \Omega \cdot \tilde{t}. \tag{27}$$

In [2] it is proved that matrix K must be full row rank; because here $K = -I_n$, the condition is fulfilled. In [2], the observer designer must include in the design procedure a subroutine for obtaining a full row rank matrix K; the procedure is easier here due to the form of the matrix K ($K = I_n$). The identity matrix will be always full row rank and, therefore, no other existence condition must be fulfilled; on the other hand, no subroutine is needed. Thus, with respect to the observer in [2], our observer is characterized by simplicity and a smaller number of constraints.

The solution of (27) is equivalent to the determination of the columns $(r+1)$, $(r+2)$, . . . , n of the matrix T. The other columns $\left(t_j,\ j=\overline{1,r}\right)$, in this first stage of the system's solution, are chosen arbitrarily [2]. After the calculation of the matrix T, we easily determine, from the second equation of (22), the matrix N as follows:

$$N=(M-KT)\begin{bmatrix} I_r \\ 0_{(n-r)\times r}\end{bmatrix}C_1^{-1}=(M+T)\begin{bmatrix} I_r \\ 0_{(n-r)\times r}\end{bmatrix}C_1^{-1}; \tag{28}$$

matrix $M\in\mathcal{R}^{n\times n}$ is random. In [2] it is also proved that a necessary condition is that the number of rows for matrix T is at least equal with the number of rows for matrix N. Here, this constraint is fulfilled ($n\geq n$).

There is no guarantee that the matrix T, determined only by using the third equation (22), verifies the other two equations (22); therefore, to overcome this drawback, we write the first equation (22) as:

$$P\begin{bmatrix} C_1 & 0_{r\times(n-r)}\end{bmatrix}=-TE, \tag{29}$$

which, by right multiplication with $\begin{bmatrix} I_r \\ 0_{(n-r)\times r}\end{bmatrix}$, leads to:

$$PC_1=-TE\begin{bmatrix} I_r \\ 0_{(n-r)\times r}\end{bmatrix}\Leftrightarrow P=-TE\begin{bmatrix} I_r \\ 0_{(n-r)\times r}\end{bmatrix}C_1^{-1}. \tag{30}$$

Thus, the calculation of the matrix T leads to the determination of the matrix P by means of (30).

Now, by right multiplication of (29) with the matrix $\begin{bmatrix} 0_{r\times(n-r)} \\ I_{n-r}\end{bmatrix}$, we get:

$$\Psi\cdot\tilde{tt}=0_{n(n-r)\times 1}, \tag{31}$$

with

$$\Psi=\begin{bmatrix} e_{1,r+1}I_n & \cdots & e_{r+1,r+1}I_n & e_{r+2,r+1}I_n & \cdots & e_{n,r+1}I_n \\ e_{1,r+2}I_n & \cdots & e_{r+1,r+2}I_n & e_{r+2,r+2}I_n & \cdots & e_{n,r+2}I_n \\ & \vdots & & & \vdots & \\ e_{1,n}I_n & \cdots & e_{r+1,n}I_n & e_{r+2,n}I_n & \cdots & e_{n,n}I_n \end{bmatrix} \tag{32}$$
$$\in\mathcal{R}^{n(n-r)\times n^2},\ \tilde{tt}=\begin{bmatrix} t_1^T & t_2^T & \cdots & t_n^T\end{bmatrix}^T\in\mathcal{R}^{n^2\times 1};$$

$e_{j,k},\ j=\overline{1,n},\ k=\overline{r+1,n}$, represent the elements of matrix E.

The second equation of (22) can be written as [2]:

$$\Theta\cdot\tilde{tt}=0_{nq\times 1}, \tag{33}$$

where

$$\Theta=\begin{bmatrix} d_{1,1}I_n & d_{2,1}I_n & \cdots & d_{n,1}I_n \\ d_{1,2}I_n & d_{2,2}I_n & \cdots & d_{n,2}I_n \\ \vdots & \vdots & \vdots & \vdots \\ d_{1,q}I_n & d_{2,q}I_n & \cdots & d_{n,q}I_n \end{bmatrix}_{nq\times n^2}; \tag{34}$$

$d_{j,k},\ j=\overline{1,n},\ k=\overline{1,q}$ represent the elements of matrix D. Putting (27) under the form [2]:

$$\begin{bmatrix} 0_{n(n-r)\times nr} & \Omega\end{bmatrix}\tilde{tt}=\tilde{m}, \tag{35}$$

and bringing together (31), (33), and (35), we get:

$$\begin{cases} \begin{bmatrix} 0_{n(n-r)\times nr} & \Omega\end{bmatrix}\tilde{tt}=\tilde{m}, \\ \Psi\cdot\tilde{tt}=0_{n(n-r)\times 1}, \\ \Theta\cdot\tilde{tt}=0_{nq\times 1} \end{cases} \tag{36}$$

or, by using the notation:

$$\pi=\begin{bmatrix} \begin{bmatrix} 0_{n(n-r)\times nr} & \Omega\end{bmatrix} \\ \Psi \\ \Theta \end{bmatrix}_{n^2\times n^2},\ \eta=\begin{bmatrix} \tilde{m} \\ 0_{n(n-r)\times 1} \\ 0_{nq\times 1}\end{bmatrix}_{n^2\times 1}, \tag{37}$$

it results the system:

$$\pi\cdot\tilde{tt}=\eta. \tag{38}$$

The previous system has n^2 unknowns (rows' number of the matrix $\tilde{tt}$) and $2n(n-r)+nq$ equations (rows' number of the matrix π); it is also demonstrated in [2] that system (38) can be solved if matrix π has full row rank and, moreover,

$$2n(n-r)+nq=n^2\Leftrightarrow r=(n+q)/2. \tag{39}$$

If r verifies (39) and π is a full row rank matrix, it results that matrix π is invertible; thus, the solution of (38) is:

$$\tilde{tt}=\pi^{-1}\eta. \tag{40}$$

Having in mind the second equation of (1), r will represent the number of sensors needed in the system. After the calculation of the vector $\tilde{tt}$, we determine $t_j,\ j=\overline{1,n}-$ the columns of matrix T by using the second equation of (32). This is then used to the calculation of matrix P, (30), and N, (28); the matrix $M\in\mathcal{R}^{n\times n}$ has been randomly chosen. If we know the matrices M, N, P for each eigenvalue λ_i of matrix Q, *i.e.* M_i, N_i, and P_i, by means of (13), we determine the row vectors h_i, $\overline{s}_i$, and $\overline{l}_i$, $i=\overline{1,n}$, we built the matrices H, $\overline{S}$, $\overline{L}$, and Λ (10), and then, by using (17) and (18), the matrices S, L, Q, and G are calculated. The observer (2) is now completely determined. As a conclusion, we modified the method presented in [2] and we adapted it to our case; thus, we managed to extend the method presented in [2] to the general class of unknown input observers; therefore, the generality character of our design procedure has been increased.

III. EXISTENCE CONDITIONS OF THE OBSERVER

The first condition for solving the system (38) (π is a full row rank matrix) is discussed in detail in [2] where is demonstrated that π is a full row rank matrix if and only if the following conditions are fulfilled:

1. $C\bar{E}$ is a full rank matrix, where

$$\bar{E} = [e_{r+1} \quad e_{r+2} \quad \cdots \quad e_n] \in \mathcal{R}^{n\times(n-r)}; \tag{41}$$

2. CD is a full column rank matrix $\Leftrightarrow$rank(CD) = q;
3. Matrix $[E_{12} \ D_1]$ is full column rank, i.e. rank$[E_{12} \ D_1] = r = (n+q)/2$, where $E_{12} \in \mathcal{R}^{r\times(n-r)}$ and $D_1 \in \mathcal{R}^{r\times q}$ are submatrices of the matrices $E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$ and $D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$, respectively.

Because the obtained observer is a combination of the observers presented in [1] and [2], the existence conditions of the new observer will be the existence conditions' reunion of the two observers. Thus, the conditions 1, 2, and 3 (constraints of the Trinh & Ha observer) are necessary for the solving of the equation system (38) and for the determination of the matrices $T_i, i=\overline{1,n}$, while the existence conditions of the Lin & Wang observer [1] are: (i) rank(CD) = rank(D); (ii) $\text{rank}\begin{bmatrix} (A-\lambda I) & D \\ C & 0 \end{bmatrix} = n + \text{rank}(D), (\forall)\lambda \in \mathrm{C}$.

The conditions 2 and (i) lead to rank(CD) = rank(D) = q; the constraints 1, 3, and (ii) remain valid for the new observer. Thus, our new observer has four existence conditions: (a) rank(CD) = rank(D) = q; (b) rank$[E_{12} \ D_1] = r$; (c) $\text{rank}\begin{bmatrix} (A-\lambda I) & D \\ C & 0 \end{bmatrix} = n+q, (\forall)\lambda \in \mathrm{C}$; (d) $\text{rank}(C\bar{E}) = \max$, matrix $\bar{E}$ having the form (41).

Remark 2. For the majority of the existing approaches, the number of unknown inputs (q) must be less than the number of outputs (r), and, moreover, additional structural requirements on the system to be observed are met [1,2,10]. Those conditions turn out to be rather restrictive because, for instance, they cannot cover the simplest class of mechanical systems with unknown inputs wherein only the position is measurable [31]. These conditions have been relaxed in only a few papers [11]; in our approach, the drawback related to the number of system inputs and outputs does not appear, this being an advantage of our method.

IV. OBSERVER ROBUSTNESS IMPROVEMENT

The advantage of choosing fast poles for the observer is that the observer estimation error e decays rapidly, but this can be achieved if and only if the system is characterized by perfect sensors and/or noise free environment. An alternative is the choice of slow poles for the observer; in this situation, the system becomes less sensitive to process disturbances and measurement noise, but the estimation error of the observer decays slowly. Thus, the observer convergence speed may be chosen by its designer with respect to its utility.

4.1 The first method for the robustness improvement

The choice of the eigenvalues λ_i influences both the convergence time and the observer error. In other words, it is possible that the choice of the observer's poles (λ_i) leads to the unwanted situation when $\lim_{t\to\infty} e(t) \neq 0$. This phenomenon is called the eigenvalues' sensitivity and is briefly analyzed in [1]. It has been proved in [32] that the sensitivity associated to the eigenvalues λ_i is measured by means of the variables $c_i = \frac{\|h_i\|_2 \cdot \|v_i\|_2}{|h_i v_i|} \geq 1$; to increase the observer robustness, we have to minimize the cost function $J_1 = \sum_{i=1}^{n} c_i$. The matrices A, B, and D from equations (1) sometimes are not completely known; thus, some small unknown variances (ΔA, ΔB, ΔD) can appear. In this case, the system can be put under another form—similar with the one described by equations (1); therefore, the robustness of the above designed observer must be improved not only from the eigenvalues' sensitivity perspective, but even from the point of view of the unknown small variances such that the negative effect of the unknown matrices ΔA, ΔB, and ΔD on the system state's estimation is minimized, if not canceled. The appearance of the matrices ΔA, ΔB, and ΔD in the system's dynamics is due to the appearance of the faults, calculation errors, modeling errors, small variance of the system in the linearization process, *etc.* [1].

If we take into consideration the uncertain matrices (ΔA, ΔB, ΔD), the dynamics of the error is [1]:

$$\dot{e}(t) = Qe(t) + (SC - I)[\Delta A\, x(t) + \Delta B\, u(t) + \Delta D\, v(t)]. \tag{42}$$

If $\Delta A = \Delta B = \Delta D = 0$, the equation (3) is obtained; otherwise, the effect of the completely unknown matrices ΔA, ΔB, and ΔD can not be completely eliminated but only minimized. Therefore, an efficient method can be the minimization of the matrix (SC-I) Frobenius norm, *i.e.* the minimization of cost function $J_2 = \|SC - \mathrm{I}\|_F$, where $\|SC - I\|_F^2 = \mathrm{trace}\left\{(SC-I)^T(SC-I)\right\}$. Thus, if both the observer poles' sensitivity and the appearance of the uncertain matrices (possibly cased by faults within the system) are taken into account, the robustness of the observer (2) may be improved by minimization of the objective function [1]:

$$J = J_1 + J_2 = \sum_{i=1}^{n} c_i + \|SC - I\|_F. \tag{43}$$

The minimization of J involves the solving of an optimal problem with respect to the row vectors $f_i, i = \overline{1,n}$. Because M_i, N_i, and P_i are determined by using the method in [2], the variation of the left eigenvectors (h_i) and right eigenvectors (v_i) depend on the choice of the row vectors $f_i, i = \overline{1,n}$, (13). In other words, h_i and v_i depend of f_i, thus J_1 depends on f_i. The minimization of the function J_1 is known in the specialty literature as the eigenvalue sensitivity reduction; this problem is equivalent with the minimization of the condition numbers associated to a matrix eigenvalues (the reciprocals of the cosines of the angles between the left and right eigenvectors); for solving this problem, the algorithms proposed in [33–35] can be used. J_2 will be minimized with respect to matrix S, which, according to (17) and (13), also depends on the row vectors $f_i, i = \overline{1,n}$.

4.2 The second method for the robustness improvement

An alternative procedure for minimizing the objective function J_2 is the determination of the matrix S (vectors f_i) such that $SC = I_n$; the obtained vectors f_i must verify the minimization condition of J_1. To write the equation $SC = I_n$ under another form (an explicit form with respect to f_i), both members of $SC = I_n$ are multiplied by matrix H; it yields:

$$\underbrace{HS}_{\bar{S}} C = H \Leftrightarrow \begin{bmatrix} \bar{s}_1 \\ \bar{s}_2 \\ \vdots \\ \bar{s}_n \end{bmatrix} C = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \Leftrightarrow \begin{bmatrix} f_1 N_1 \\ f_2 N_2 \\ \vdots \\ f_n N_n \end{bmatrix} C = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \tag{44}$$
$$\Leftrightarrow f_i N_i C = h_i, i = \overline{1,n}$$

or, taking into account the first equation of (13), we get:

$$f_i (N_i C - M_i) = 0, i = \overline{1,n}. \tag{45}$$

According to (21) and the third equation of (20), $N_i C - M_i = T_i, i = \overline{1,n}$; in these circumstances, (45) becomes:

$$f_i T_i = 0, i = \overline{1,n}. \tag{46}$$

The solution of the previous equation makes an analogy with the solution of the equations $T_i^T \tilde{f}_i = 0, T_i^T$ being known matrices, while $\tilde{f}_i -$ is unknown vectors; in this case, $\tilde{f}_i$ belongs to the null space associated to the square matrix T_i^T. The solution of equation (46) depends on $\tilde{f}_i$. Thus, f_i is called the left null vector associated to the matrix T_i and can be determined by using the equation $f_i = \tilde{f}_i^H = \mathrm{conj}\left(\tilde{f}_i^T\right)$.

4.3 The novel method for the robustness improvement

An alternative to the above presented methods for the observer robustness improvement with respect to uncertain matrices ΔA, ΔB, ΔD belongs to the authors of this paper and is presented below. The method is based on the addition of three terms in the observer equations. The observer equations become:

$$\begin{cases} \dot{z}(t) = Qz(t) + Ly(t) + Gu(t) + \alpha_A + \alpha_B + \alpha_D, \\ x(t) = z(t) + Sy(t); \end{cases} \tag{47}$$

the robustness terms (α_A, α_B, α_D), added in the observer equations, must compensate the errors caused by uncertain matrices of the system; the determination of the three terms is made by using the Lyapunov theory. This approach replaces the difficult process of the objective function J_2 minimization or the solving of the equation (46). The expressions of the robustness terms are presented within the following theorem.

Theorem 2. If there exists a matrix $\tilde{P} > 0$ and positive scalars β_1, β_2, β_3, β_4, $\beta_A = \beta_1(1 - \beta_4^{-1})$ satisfying the constraint:

$$\begin{bmatrix} Q^T\tilde{P} + \tilde{P}Q + \beta_A \delta_A^2 I & \tilde{P} \\ \tilde{P} & -\left(\beta_1^{-1} + \beta_2^{-1} + \beta_3^{-1}\right)^{-1} I \end{bmatrix} < 0, \tag{48}$$

the state estimation error of the observer (2) asymptotically converges to zero if

$$\alpha_A = \begin{cases} \beta_1(\beta_4 - 1)\delta_A^2 \dfrac{\hat{x}^T\hat{x}}{2\bar{r}^T\bar{r}} \tilde{P}^{-1}C^T\bar{r}, & \bar{r} \neq 0 \\ 0, \quad \bar{r} = 0 \end{cases},$$
$$\alpha_B = \begin{cases} -\dfrac{\beta_2 \rho_u^2 \delta_B^2}{2\bar{r}^T\bar{r}} \tilde{P}^{-1}C^T\bar{r}, & \bar{r} \neq 0 \\ 0, \quad \bar{r} = 0 \end{cases}, \tag{49}$$
$$\alpha_D = \begin{cases} -\dfrac{\beta_3 \rho_v^2 \delta_D^2}{2\bar{r}^T\bar{r}} \tilde{P}^{-1}C^T\bar{r}, & \bar{r} \neq 0 \\ 0, \quad \bar{r} = 0 \end{cases},$$

where $\bar{r} = \hat{y} - y = Ce$, while the time variant matrices $\overline{\Delta A} = (SC - I)\Delta A, \overline{\Delta B} = (SC - I)\Delta B, \overline{\Delta D} = (SC - I)\Delta D$, the known and the unknown inputs (u and v) are bounded, *i.e.*:

$$\left\|\overline{\Delta A}\right\| < \delta_A, \left\|\overline{\Delta B}\right\| < \delta_B, \left\|\overline{\Delta D}\right\| < \delta_D, \|u\| < \rho_u, \|v\| < \rho_v. \tag{50}$$

Proof. In the proof of Theorem 2, we use the inequality [13]:

$$X^T Y + Y^T X \le \beta X^T X + \beta^{-1} Y^T Y, \tag{51}$$

for any matrices X and Y, and the constant $\beta > 0$.

Considering the Lyapunov function $V(e) = e^T \tilde{P} e$, we get

$$\dot{V} = e^T\left(Q^T\tilde{P} + \tilde{P}Q\right)e + \underbrace{\left(x^T\overline{\Delta A}^T\tilde{P}e + e^T\tilde{P}\overline{\Delta A}\,x\right)}_{S_1} + \underbrace{\left(u^T\overline{\Delta B}^T\tilde{P}e + e^T\tilde{P}\overline{\Delta B}\,u\right)}_{S_2} + \underbrace{\left(v^T\overline{\Delta D}^T\tilde{P}e + e^T\tilde{P}\overline{\Delta D}\,v\right)}_{S_3} + 2\alpha_A^T\tilde{P}e + 2\alpha_B^T\tilde{P}e + 2\alpha_D^T\tilde{P}e, \tag{52}$$

where $\overline{\Delta A} = (SC - I)\Delta A, \overline{\Delta B} = (SC - I)\Delta B, \overline{\Delta D} = (SC - I)\Delta D$; to obtain the expressions (52), we had in mind the equations: $\alpha_A^T\tilde{P}e + e^T\tilde{P}\alpha_A = 2\alpha_A^T\tilde{P}e, \alpha_B^T\tilde{P}e + e^T\tilde{P}\alpha_B = 2\alpha_B^T\tilde{P}e, \alpha_D^T\tilde{P}e + e^T\tilde{P}\alpha_D = 2\alpha_D^T\tilde{P}e$.

By means of the inequality (51), written for the sets:

$$\begin{aligned}(\Sigma_1)&:\left\{X = \overline{\Delta A}\,x \quad Y = \tilde{P}e \quad \beta = \beta_1\right\};\\ (\Sigma_2)&:\left\{X = \overline{\Delta B}\,x \quad Y = \tilde{P}e \quad \beta = \beta_2\right\};\\ (\Sigma_3)&:\left\{X = \overline{\Delta D}\,x \quad Y = \tilde{P}e \quad \beta = \beta_3\right\},\end{aligned} \tag{53}$$

the sum $S_{123} = S_1 + S_2 + S_3$ in equation (52) becomes:

$$S_{123} \le \beta_1 x^T\overline{\Delta A}^T\overline{\Delta A}\,x + \beta_2 u^T\overline{\Delta B}^T\overline{\Delta B}\,u + \beta_3 v^T\overline{\Delta D}^T\overline{\Delta D}\,v + \left(\beta_1^{-1} + \beta_2^{-1} + \beta_3^{-1}\right)e^T\tilde{P}^2 e; \tag{54}$$

using (54), $\dot{V}$ gets the form:

$$\dot{V} \le e^T\left[Q^T\tilde{P} + \tilde{P}Q + \left(\beta_1^{-1} + \beta_2^{-1} + \beta_3^{-1}\right)\tilde{P}^2\right]e + 2\left(\alpha_A^T + \alpha_B^T + \alpha_D^T\right)\tilde{P}e + \beta_1 x^T\overline{\Delta A}^T\overline{\Delta A}\,x + \beta_2 u^T\overline{\Delta B}^T\overline{\Delta B}\,u + \beta_3 v^T\overline{\Delta D}^T\overline{\Delta D}\,v. \tag{55}$$

If the assumptions (50) are taken into account, replacing the above state x with $\hat{x} - e$, we can demonstrate the inequalities

$$\begin{aligned}&\beta_1 x^T\overline{\Delta A}^T\overline{\Delta A}\,x \le \beta_1\delta_A^2\left(\hat{x}^T\hat{x} + e^Te\right) - \beta_1\delta_A^2\left(\hat{x}^Te + e^T\hat{x}\right),\\ &\beta_2 u^T\overline{\Delta B}^T\overline{\Delta B}\,u \le \beta_2\delta_B^2\rho_u^2, \beta_3 v^T\overline{\Delta D}^T\overline{\Delta D}\,v \le \beta_3\delta_D^2\rho_v^2\end{aligned} \tag{56}$$

and the inequality (55) becomes

$$\dot{V} \le e^T\left[Q^T\tilde{P} + \tilde{P}Q + \left(\beta_1^{-1} + \beta_2^{-1} + \beta_3^{-1}\right)\tilde{P}^2\right]e + 2\left(\alpha_A^T + \alpha_B^T + \alpha_D^T\right)\tilde{P}e + \beta_1\delta_A^2\left(\hat{x}^T\hat{x} + e^Te\right) - \beta_1\delta_A^2\left(\hat{x}^Te + e^T\hat{x}\right) + \beta_2\delta_B^2\rho_u^2 + \beta_3\delta_D^2\rho_v^2. \tag{57}$$

Now, using again the inequality (51), written for the set $(\Sigma_4):\{X = \hat{x} \quad Y = e \quad \beta = \beta_4\}$, it yields:

$$\hat{x}^Te + e^T\hat{x} \le \beta_4\hat{x}^T\hat{x} + \beta_4^{-1}e^Te. \tag{58}$$

By means of inequality (58), the inequality (57) becomes:

$$\dot{V} \le e^T\left[Q^T\tilde{P} + \tilde{P}Q + \left(\beta_1^{-1} + \beta_2^{-1} + \beta_3^{-1}\right)\tilde{P}^2 + \beta_A\delta_A^2 I\right]e + 2\left(\alpha_A^T + \alpha_B^T + \alpha_D^T\right)\tilde{P}e + \beta_1\delta_A^2(1 - \beta_4)\hat{x}^T\hat{x} + \beta_2\delta_B^2\rho_u^2 + \beta_3\delta_D^2\rho_v^2. \tag{59}$$

where $\beta_A = \beta_1\left(1 - \beta_4^{-1}\right)$. If we choose

$$\begin{aligned}&2\alpha_A^T\tilde{P}e = -\beta_1\delta_A^2(1 - \beta_4)\hat{x}^T\hat{x}, 2\alpha_B^T\tilde{P}e = -\beta_2\delta_B^2\rho_u^2,\\ &2\alpha_D^T\tilde{P}e = -\beta_3\delta_D^2\rho_v^2,\end{aligned} \tag{60}$$

the inequality (59) attains the form:

$$\dot{V} \le e^T\left[Q^T\tilde{P} + \tilde{P}Q + \left(\beta_1^{-1} + \beta_2^{-1} + \beta_3^{-1}\right)\tilde{P}^2 + \beta_A\delta_A^2 I\right]e. \tag{61}$$

For a positive defined matrix $\tilde{P}$ calculated such that $Q^T\tilde{P} + \tilde{P}Q + \left(\beta_1^{-1} + \beta_2^{-1} + \beta_3^{-1}\right)\tilde{P}^2 + \beta_A\delta_A^2 I < 0$, the inequality (61) is $\dot{V} < 0$ which corresponds to a stable system. To solve the above nonlinear matrix inequality, we can use the Schur complement theorem to transform it into a linear matrix inequality (LMI); thus, it results in

$$\begin{bmatrix} Q^T\tilde{P} + \tilde{P}Q + \beta_A\delta_A^2 I & \tilde{P} \\ \tilde{P} & -\left(\beta_1^{-1} + \beta_2^{-1} + \beta_3^{-1}\right)^{-1} I \end{bmatrix} < 0. \tag{62}$$

The robustness terms are deduced by solving (60); thus, the first equation (60) can be written as: $2\alpha_A^T\tilde{P}e = \beta_1\delta_A^2(\beta_4 - 1)\hat{x}^T\hat{x} \Leftrightarrow \alpha_A = \dfrac{\beta_1\delta_A^2(\beta_4 - 1)}{2}\hat{x}^T\hat{x}\tilde{P}^{-1}\left(e^{-1}\right)^T$; using the notation $\overline{r} = \hat{y} - y = Ce$, it can be shown that $\left(e^{-1}\right)^T = \left(C^T\overline{r}\right)/\left(\overline{r}^T\overline{r}\right)$ and, for $\overline{r} \ne 0$, it results: $\alpha_A = \beta_1(\beta_4 - 1)\delta_A^2\dfrac{\hat{x}^T\hat{x}}{2\overline{r}^T\overline{r}}\tilde{P}^{-1}C^T\overline{r}$; if $\overline{r} = 0$, one gets $\alpha_A = 0$. The first equation of (49) is now completely demonstrated; also, the expressions of the robustness terms α_B and α_D are deduced. The proof of Theorem 2 is now complete.

The introduction of the robustness terms represents one of the main contribution of the paper with respect to the works [1,2]. Actually, in [2], the possible system's uncertainties are not taken into account and, of course, this is one of the paper's disadvantages; this drawback is overcome here.

Remark 3. If we take into consideration only the uncertainties, the advantage of the above presented method is related to the fact that there is no need to minimize the objective function J_1 or to solve equation (46) with the unknowns f_i. Therefore, these vectors can be randomly chosen and the observer state estimation error converges asymptotically to zero. If we consider both the eigenvalues' sensitivity and the uncertainties, we must determine the vectors f_i by minimization of the objective function J_1 and, after that, instead of the minimization of the second objective function (J_2), we improve the system robustness by calculation of the three robustness terms. A problem which can appear in the observer implementation process is related to the case when, although $\bar{r} \to 0$, the robustness terms α_A, α_B, and α_D are not bounded. In [13], this problem is solved as follows: α_A, α_B, and α_D are null if $\|\bar{r}\| < \varepsilon$, where $\varepsilon > 0$ is a constant having small values, chosen by the observer designer.

V. DESIGN ALGORITHM

Step 1. We choose λ_i – the eigenvalues of the matrix Q, we calculate the matrices $E_i = A - \lambda_i I$, and $\bar{E}$ (equation (41)); after that, the matrices E and D are partitioned with respect to the existence condition 3). We calculate r by means of (39), rank(CD), rank(D), rank[E_{12} D_1], $\text{rank}(C\bar{E})$, $\text{rank}\begin{bmatrix}(A-\lambda I) & D \\ C & 0\end{bmatrix}$, $(\forall)\lambda \in \mathbb{C}$, and we verify if the observer constraints (conditions a), b), c), and d)) are fulfilled.
Step 2. For each $i = \overline{1, n}$, we randomly choose a matrix M and we calculate $\tilde{m}, \Omega, \Psi$ (equation (32)), Θ (equation (34)), π and η (equation (37)), and the matriceal equation (38) is solved. We calculate vector $t\tilde{t}$ (equation (32)), and we determine t_j, $j = \overline{1, n}$ – the column of the matrix T, and hence the matrix T.
Step 3. We calculate the matrices P and N (using equation (28) and (30)), and for each λ_i, we verify if the conditions (21) are valid.
Step 4. We use the novel method for robustness improvement; thus, if eigenvalues' sensitivity is not taken into account, the vectors f_i are random; otherwise, these vectors are calculated by minimization of the ojbective function J_1. Then, with (13), we calculate h_i, $\bar{s}_i$ and $\bar{l}_i$, $i = \overline{1, n}$.
Step 5. Using h_i, $\bar{s}_i$, and $\bar{l}_i$, we form the matrices H, $\bar{S}$, $\bar{L}$ and Λ – equations (10). This stage is the last one within the "for" cycle; this cycle contains the algorithm first 5 steps.
Step 6. We calculate the matrices S, L, Q, and G by means of equations (17) and (18) and we check the simultaneous fulfillment of the conditions (4).
Step 7. The positive constants β_1, β_2, β_3, β_4, the bounds δ_A, δ_B, δ_D, ρ_u, ρ_v are chosen and we calculate β_A; we solve LMI (62) and we obtain $\tilde{P}$; then, we calculate the three robustness terms (equations (49)).
Step 8. Using now the matrices S, L, Q, and G, we design the observer described by equations (2)—the case without uncertainties and, if we do not obtain $\|\bar{r}\| < \varepsilon$, we use the observer described by equations (47)—the case of the dynamics with uncertainties. This way, we achive the main objective of our design procedure: the system state reconstruction (determination of the state $\hat{x}(t)$).

VI. NUMERICAL SIMULATION RESULTS

6.1 Numerical simulation setup

The validation of our new algorithm for an observer design is performed, in this section, in the MATLAB/SIMULINK environment, for the case of lateral motion of a light aircraft [36]. Aircraft

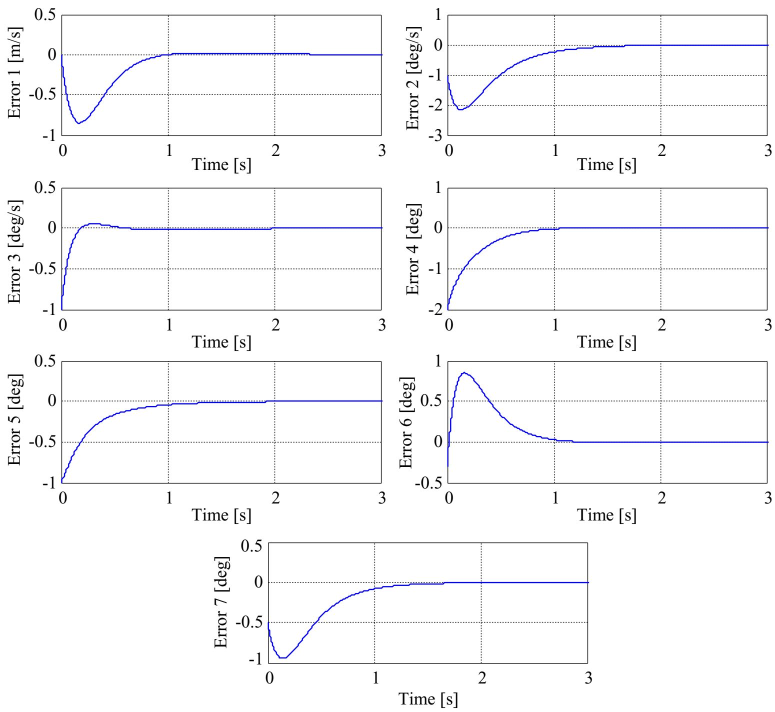

Fig. 1. Observer estimation errors—Case 1: dynamics without uncertainties.

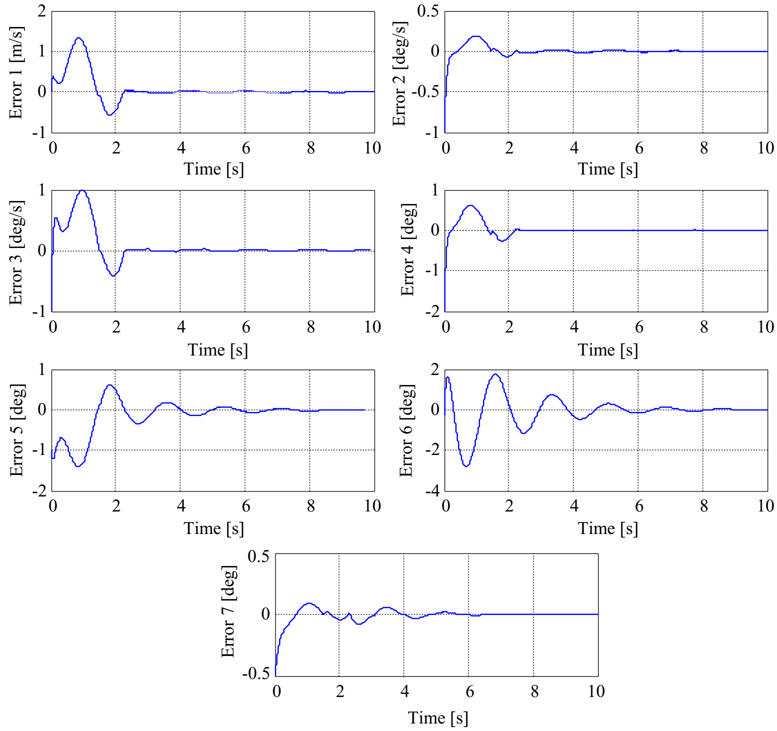

Fig. 2. Observer estimation errors—Case 2: dynamics with uncertainties.

flight is often influenced by disturbances like longitudinal or vertical wind shears, atmospheric turbulences, or errors of the sensors. From the aircraft dynamics' point of view, these represent unknown inputs. Here, we validate our new observer for the case of an aircraft flight but the observer may be used, with good results, in any other examples due to its generality character. The aircraft dynamics has the form (1) with [36]:

$$x=[V_y \quad \omega_x \quad \omega_z \quad \phi \quad \psi \quad \delta_r \quad \delta_a]^T,\ y=[V_y \quad \omega_x \quad \omega_z]^T,$$
$$u=[\delta_{r_c} \quad \delta_{a_c}]^T, C=[I_r \quad 0_{r\times(n-r)}],$$
$$A=\begin{bmatrix} -0.30 & 0 & -33 & 9.81 & 0 & -5.4 & 0 \\ -0.1 & -8.3 & 3.75 & 0 & 0 & 0 & -28.6 \\ 0.37 & 0 & -0.64 & 0 & 0 & -9.5 & 0 \\ 0 & 1 & 0.01 & 0 & 0.01 & 0 & 0 \\ 0 & 0 & 1 & 0.001 & 0 & 0.001 & 0 \\ 0 & 0 & -0.01 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0.01 & 0 & -0.001 & 0 & -5 \end{bmatrix}, \tag{63}$$
$$B=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -0.01 \\ 20 & 0 \\ 0 & 20 \end{bmatrix}, D=\begin{bmatrix} 0 \\ 0 \\ -0.01 \\ 0 \\ 0 \\ -20 \\ 0 \end{bmatrix};$$

here, V_y is the aircraft lateral velocity, ω_x − aircraft roll angular rate, ω_z − aircraft yaw angular rate, φ − aircraft roll angle, ψ − aircraft yaw angle, δ_r and δ_a− deflections of the rudder and ailerons, respectively, while δ_{r_c} and δ_{a_c} are the commands applied by the pilot to the rudder and ailerons, respectively. Using the matrices from (64) we obtained: $n = 7$, $m = 2$, $q = 1$, and $r = 4$. The observer validation is performed both for the dynamics with uncertainties (ΔA, ΔB, $\Delta D \neq 0$) and without uncertainties (ΔA, ΔB, $\Delta D = 0$) without taking into consideration the eigenvalues' sensitivity.

6.2 Results and discussion

For increasing the observer robustness, we use the new technique with the adding of the three robustness terms in the observer equations; therefore, as we have already proved in the previous sections of the paper, the vectors f_i can be randomly chosen. The eigenvalues of matrix Q (chosen in the left-hand side of the complex plane) are on the main diagonal of matrix $\Lambda = \text{diag}\{-2 \ -4 \ -6 \ -8 \ -10 \ -12 \ -14\}$. These eigenvalues must verify the Perron-Frobenius theorem. The solving of the linear matriceal inequality (62) is performed with respect to the unknown matrix $\tilde{P}$, by using the MATLAB/SIMULINK LMI tool; we have chosen $\beta_1 = \beta_2 = \beta_3 = 0.1$, and $\beta_4 = 0.95$. Step 7 of the above algorithm is achieved only for the case of the dynamics with uncertainties (ΔA, ΔB, $\Delta D \neq 0$).

In Fig. 1 we present the time histories of the state estimation errors $e_i(t) = \hat{x}_i(t) - x_i(t), i = \overline{1,7}$ for the first simulation case (ΔA, ΔB, $\Delta D = 0$) while in Fig. 2 we represent the time histories of same estimation errors but for the second simulation case (ΔA, ΔB, $\Delta D \neq 0$) From these two figures one remarks the cancellation of the estimation errors, hence a proper functioning of the new designed observer. The cancellation of the estimation error (the difference between the real state vector and the estimated one) is equivalent with the achievement of the state reconstruction $(\hat{x} \to x)$. The observer convergence speed represents a good convergence speed in the research area of the multivariable systems with unknown inputs and, therefore, we conclude that the new designed observer can be used with very good results to the state estimation in the case of multivariable systems with unknown inputs. The observer overshoot and convergence speed can be improved by modifying the design constants β_1, β_2, β_3, β_4 and ε (here, we have chosen $\varepsilon = 0.05$). The observer estimation error e decays rapidly if fast poles for the observer are chosen and decays slowly if the observer has slow poles; on the other hand, fast poles can be selected if the system is characterized by perfect sensors and/or noise free environment, while the only advantage of the slow poles' choice is related to the fact that the system becomes less sensitive to process disturbances and measurement noise.

VII. CONCLUSIONS

In this paper we design a new approach for the state estimation problem in the case of linear multivariable systems with unknown inputs. The design method is a mixture of the design approaches presented in [1] and [2]; moreover, to cancel the possible system's uncertainties and to improve the design approach of the observer, some robustness terms and new calculation techniques have been added in the observer design. The attractive feature of the proposed observer is its simplicity. Our observer has been validated by numerical simulation for the case of an aircraft lateral motion; the results are very good: the observer errors tend to zero even in the case of systems with uncertainties. The new design procedure can be extended in the future and a new observer with unknown inputs' reconstruction can be designed; it may be a subsystem (fault detection/diagnosis scheme) of a typical fault-tolerant control system.

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